4. Sequences

4.1 CATEGORIES

When first meeting a new class of mathematical objects – such as topological spaces or abelian groups – it is natural to try and learn about the underlying structures (open sets, commutative multiplication laws) which characterize each object, and to carefully describe a reasonable notion of functions (continuous maps, group homomorphisms) which preserve that structure. In almost all cases of interest, it turns out that composing two such structure-preserving functions produces another such function. The following definition provides a convenient umbrella under which such (structure, function) pairs reside.

DEFINITION 4.1. A **category** $\mathscr{C}$ consists of

(1) a collection $\mathscr{C}_0$ whose elements are called *objects*,
(2) for every pair of objects x, y in $\mathscr{C}$ a set $\mathscr{C}(x,y)$ of *morphisms* from x to y, whose elements we denote $f, g, \ldots : x \to y$, and
(3) for each triple x, y, z of objects a *composition law* $\mathscr{C}(x,y) \times \mathscr{C}(y,z) \to \mathscr{C}(x,z)$ sending $f : x \to y$ and $g : y \to z$ to some $g \circ f : x \to z$,

subject to the **identity** and **associativity** axioms.

Our definition remains incomplete until we spell out these two axioms; here they are:

(1) for each x in $\mathscr{C}_0$, there is a distinguished *identity* morphism $1_x : x \to x$ satisfying both
$$g \circ 1_x = g \text{ and } 1_x \circ h = h$$
for any object y and morphisms $g : y \to x$ and $h : x \to y$;
(2) given any triple of morphisms of the form
$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w,$$
the *associativity* condition $h \circ (g \circ f) = (h \circ g) \circ f$ holds.

Instances of (object, morphism) pairs in mathematics which satisfy these two axioms are ubiquitous — consider, for instance:

- the category **Set** of (sets, functions),
- the category **Grp** of (groups, group homomorphisms),
- its subcategory **AbGrp** of (abelian groups, abelian group homomorphisms),
- the category **SC** of (simplicial complexes, simplicial maps),
- the category $\mathbf{Vect}_{\mathbb{F}}$ of ($\mathbb{F}$-vector spaces, $\mathbb{F}$-linear maps) over a field $\mathbb{F}$, etc.

One can encode the associativity axiom in the form of a *commuting square*, like so:

$$\begin{array}{ccc}
\mathscr{C}(x,y) \times \mathscr{C}(y,z) \times \mathscr{C}(z,w) & \xrightarrow{(f,g,h) \mapsto (g \circ f, h)} & \mathscr{C}(x,z) \times \mathscr{C}(z,w) \\
\Big\downarrow {\scriptstyle (f,g,h) \mapsto (f, h \circ g)} & & \Big\downarrow {\scriptstyle (g \circ f, h) \mapsto h \circ (g \circ f)} \\
\mathscr{C}(x,y) \times \mathscr{C}(y,w) & \xrightarrow[(f, h \circ g) \mapsto (h \circ g) \circ f]{} & \mathscr{C}(x,w)
\end{array}$$

At first glance, this diagrammatic translation of $(h \circ g) \circ f = h \circ (g \circ f)$ might come across as an elaborate crime against brevity. There are, however, several compelling reasons to become familiar with the language of commuting diagrams — for one thing, there are many such diagrams in our immediate future. Another special feature of the categorical philosophy, besides this profusion of commuting diagrams, is that it can be turned inwards to reason about the theory of categories itself. Those under its influence naturally ask what key piece of structure must be preserved by functions which map one category $\mathscr{C}$ to another category $\mathscr{C}'$.

DEFINITION 4.2. A **functor** $\boldsymbol{F} : \mathscr{C} \to \mathscr{C}'$ assigns

(1) to each object x in $\mathscr{C}_0$ an object $\boldsymbol{F}x$ in $\mathscr{C}'_0$, and
(2) to each morphism $f : x \to y$ in $\mathscr{C}$ a morphism $\boldsymbol{F}f : \boldsymbol{F}x \to \boldsymbol{F}y$ in $\mathscr{C}'$,

subject to the following conditions:

(1) we have $\boldsymbol{F}1_x = 1_{\boldsymbol{F}x}$ for each x in $\mathscr{C}_0$, and
(2) for any pair of morphisms f in $\mathscr{C}(x,y)$ and g in $\mathscr{C}(y,z)$, we have

$$\boldsymbol{F}(g \circ f) = \boldsymbol{F}g \circ \boldsymbol{F}f$$

(Here the composition on the left takes place in $\mathscr{C}$ while the composition on the right takes place in $\mathscr{C}'$).

Thus, a functor $\mathscr{C} \to \mathscr{C}'$ sends $\mathscr{C}$-objects to $\mathscr{C}'$-objects and the corresponding $\mathscr{C}$-morphisms to $\mathscr{C}'$-morphisms in a manner that duly respects composition laws of both $\mathscr{C}$ and $\mathscr{C}'$. One of the exercises to this Chapter asks you to define the composite of two functors and hence construct the category **Cat** containing (categories, functors). We have been discussing categories and functors because of the next result, which catalogues one of the most important properties of simplicial homology (see Section 4).

THEOREM 4.3. *For each dimension $k \geq 0$, the assignment*

$$K \mapsto \mathbf{H}_k(K;\mathbb{F})$$

constitutes a functor from the category **SC** *of simplicial complexes and maps to the category* $\mathbf{Vect}_{\mathbb{F}}$ *of vector spaces over* $\mathbb{F}$.

We already know from Chapter 3 that every simplicial complex K can be assigned a vector space $\mathbf{H}_k(K;\mathbb{F})$ by first building the simplicial chain complex $(\mathbf{C}_\bullet(K), \partial^K)$ and then extracting the relevant quotient $\ker \partial^K_k / \operatorname{img} \partial^K_{k+1}$. So the new content of Theorem 4.3 lies entirely on the level of morphisms — we must first show that every simplicial map $f : K \to L$ induces a well-defined linear map $\mathbf{H}_k f : \mathbf{H}_k(K;\mathbb{F}) \to \mathbf{H}_k(L;\mathbb{F})$ of homology groups; and next, we have to confirm that given some other simplicial map $g : L \to M$, we have an equality

$$\mathbf{H}_k(g \circ f) = \mathbf{H}_k g \circ \mathbf{H}_k f$$

of linear maps $\mathbf{H}_k(K;\mathbb{F}) \to \mathbf{H}_k(M;\mathbb{F})$. These are our goals in the next two Sections.

4.2 CHAIN MAPS

Fix simplicial complexes K and L as well we as a simplicial map $f : K \to L$ and a coefficient field $\mathbb{F}$. We will continue to write $(\mathbf{C}_\bullet(K), \partial^K_\bullet)$ to indicate the simplicial chain complex of K (and similarly for L).

DEFINITION 4.4. For each dimension $k \geq 0$, let $\mathbf{C}_k f : \mathbf{C}_k(K) \to \mathbf{C}_k(L)$ be the $\mathbb{F}$-linear map between chain groups defined by the following action on each basis k-simplex σ of K:

$$\mathbf{C}_k f(\sigma) = \begin{cases} f(\sigma) & \text{if } \dim f(\sigma) = k \\ 0 & \text{otherwise.} \end{cases}$$

Perhaps the most interesting aspect of this definition is its piecewise nature — some simplices σ are faithfully mapped onto their images $f(\sigma)$ while others are sent to zero, depending on whether f is injective on their vertices or not. This is a necessary bit of book-keeping: we want to produce a map from k-chains to k-chains for each k, and the image $f(\sigma)$ of a k-simplex σ will not be a basis element of $\mathbf{C}_k(L)$ unless $\dim f(\sigma) = k$. Our next order of business is to see how the linear maps $\mathbf{C}_\bullet f$ interact with the two boundary operators $\partial^K_\bullet$ and $\partial^L_\bullet$. It turns out that the diagram below is a commuting square (in the category $\mathbf{Vect}_\mathbb{F}$) for each $k \geq 0$:

$$\begin{array}{ccc} \mathbf{C}_k(K) & \xrightarrow{\partial^K_k} & \mathbf{C}_{k-1}(K) \\ {\scriptstyle \mathbf{C}_k f}\downarrow & & \downarrow{\scriptstyle \mathbf{C}_{k-1} f} \\ \mathbf{C}_k(L) & \xrightarrow[\partial^L_k]{} & \mathbf{C}_{k-1}(L) \end{array}$$

PROPOSITION 4.5. *For each dimension $k \geq 0$, and k-simplex σ in K, we have an equality*

$$\partial^L_k \circ \mathbf{C}_k f(\sigma) = \mathbf{C}_{k-1} f \circ \partial^K_k(\sigma).$$

PROOF. Given Definition 4.4, the argument naturally decomposes into two cases.

Case 1: $\dim f(\sigma) = k$**.** Here $\mathbf{C}_k f(\sigma) = f(\sigma)$ and we have

$$\begin{aligned} \partial^L_k \circ \mathbf{C}_k f(\sigma) &= \partial^L_k f(\sigma) && \text{since } \mathbf{C}_k f(\sigma) = f(\sigma), \\ &= \sum_{i=0}^{k} (-1)^i \cdot f(\sigma)_{-i} && \text{by Definition 3.7.} \end{aligned}$$

On the other hand, $\partial^K_k(\sigma)$ equals $\sum_{i=0}^k (-1)^i \sigma_{-i}$, and since f is injective on the vertices of σ it must also be injective on the vertices of each face σ_{-i}. Thus, $\mathbf{C}_{k-1} f(\sigma_{-i}) = f(\sigma_{-i})$ for each i, and we have

$$\begin{aligned} \mathbf{C}_{k-1} f \circ \partial^K_k(\sigma) &= \mathbf{C}_{k-1} f \left(\sum_{i=0}^{k} (-1)^i \cdot \sigma_{-i} \right) && \text{by Definition 3.7,} \\ &= \sum_{i=0}^{k} (-1)^i f(\sigma)_{-i} && \text{by Definition 4.4.} \end{aligned}$$

Thus, $\partial^L_k \circ \mathbf{C}_k f(\sigma)$ and $\mathbf{C}_{k-1} f \circ \partial^K_k(\sigma)$ coincide in this case.

Case 2: $\dim f(\sigma) < k$**.** Here $\mathbf{C}_k f(\sigma)$ equals zero by definition, and hence so does its boundary in L. It suffices to show that the other composite $\mathbf{C}_{k-1} f \circ \partial^K_k(\sigma)$ is also zero. To this end, impose orientations o_K and o_L on K and L so that f is orientation-preserving, i.e, $o_K(v) < o_K(v')$ forces $o_L(f(v)) < o_L(f(v'))$. Writing σ as an o_K-oriented simplex $(v_0, \ldots, v_k)$, we must have $f(v_p) = f(v_{p+1})$ for some p in $\{0, \ldots, k-1\}$. Thus, f fails to be injective on the vertices of every face σ_{-i} of σ, with the possible exceptions of σ_{-p} and $\sigma_{-(p+1)}$. Now,

$$\begin{aligned} \mathbf{C}_{k-1} f \circ \partial^K_k(\sigma) &= \mathbf{C}_{k-1} f \left(\sum_{i=0}^{k-1} (-1)^i \sigma_{-i} \right) && \text{by Definition 3.7} \\ &= \sum_{i=0}^{k-1} (-1)^i \mathbf{C}_{k-1} f(\sigma_{-i}) && \text{by linearity of } \mathbf{C}_k f \\ &= (-1)^p \left[f\left(\sigma_{-p}\right) - f\left(\sigma_{-(p+1)}\right) \right] && \text{by Definition 4.4.} \end{aligned}$$

Here the last step follows from our observation that $\mathbf{C}_k f$ evaluates to zero on all other other faces of $f(\sigma)$ since f will not be injective on their vertices. But now, $f(\sigma_{-p})$ and $f(\sigma_{-p+1})$ are the same simplex in L, since both have vertices $(f(v_0), \ldots, f(v_p), f(v_{p+2}), \ldots, f(v_k))$. Thus, both $\partial_k^L \circ \mathbf{C}_k f(\sigma)$ and $\mathbf{C}_{k-1} f \circ \partial_k^K(\sigma)$ equal zero in this case. □

As a consequence of this result, we are able to use the simplicial map $f : K \to L$ to produce a sequence of linear maps $\mathbf{C}_\bullet f : \mathbf{C}_\bullet K \to \mathbf{C}_\bullet L$ between the chain groups which form a ladder-shaped commuting diagram:

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{\partial_{k+1}^K} & \mathbf{C}_k(K) & \xrightarrow{\partial_k^K} & \mathbf{C}_{k-1}(K) & \xrightarrow{\partial_{k-1}^K} & \cdots & \xrightarrow{\partial_2^K} & \mathbf{C}_1(K) & \xrightarrow{\partial_1^K} & \mathbf{C}_0(K) & \xrightarrow{0} & 0 \\
 & & \downarrow{\scriptstyle \mathbf{C}_k f} & & \downarrow{\scriptstyle \mathbf{C}_{k-1} f} & & & & \downarrow{\scriptstyle \mathbf{C}_1 f} & & \downarrow{\scriptstyle \mathbf{C}_0 f} & & \| \\
\cdots & \xrightarrow[\partial_{k+1}^L]{} & \mathbf{C}_k(L) & \xrightarrow[\partial_k^L]{} & \mathbf{C}_{k-1}(L) & \xrightarrow[\partial_{k-1}^L]{} & \cdots & \xrightarrow[\partial_2^L]{} & \mathbf{C}_1(L) & \xrightarrow[\partial_1^L]{} & \mathbf{C}_0(L) & \xrightarrow[0]{} & 0
\end{array}$$

This is a standard example of a **chain map**, which can be used to relate arbitrary (i.e., not necessarily simplicial) chain complexes.

DEFINITION 4.6. A **chain map** $\phi_\bullet$ from $(C_\bullet, d_\bullet)$ to $(C'_\bullet, d'_\bullet)$ is defined to be a sequence of $\mathbb{F}$-linear maps $\{\phi_k : C_k \to C'_k \mid k \geq 0\}$ which satisfy

$$d'_k \circ \phi_k = \phi_{k-1} \circ d_k$$

for each $k \geq 0$.

Proposition 4.5 can be now be rephrased:

$$\text{simplicial maps } f : K \to L \text{ induce chain maps } \mathbf{C}_\bullet f : (\mathbf{C}_\bullet(K), \partial_\bullet^K) \to (\mathbf{C}_\bullet(L), \partial_\bullet^L).$$

It turns out that chain maps form the correct notion of morphisms in the category of chain complexes; their composition is not too difficult to define, and will be addressed by Exercise 4.3.

4.3 FUNCTORIALITY

To continue our proof of Theorem 4.3, we will use chain maps to construct maps of homology groups.

PROPOSITION 4.7. *Let $\phi_\bullet : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ be a chain map. For each dimension $k \geq 0$, there is a well-defined $\mathbb{F}$-linear map $\mathbf{H}_k\phi : \mathbf{H}_k(C_\bullet, d_\bullet) \to \mathbf{H}_k(C'_\bullet, d'_\bullet)$ induced by $\phi_\bullet$.*

PROOF. To induce a map of quotient vector spaces $\ker d_k / \operatorname{img} d_{k+1} \to \ker d'_k / \operatorname{img} d'_{k+1}$, it suffices to show that ϕ_k maps $\ker d_k$ to $\ker d'_k$ and $\operatorname{img} d_{k+1}$ to $\operatorname{img} d'_{k+1}$. First consider $\xi \in C_k$ satisfying $d_k(\xi) = 0$. Using Definition 4.6, we get

$$d'_k \circ \phi_k(\xi) = \phi_{k-1} \circ d_k(\xi) = 0,$$

so $\phi_k(\xi)$ lies in $\ker d'_k$ as desired. Next, if $\eta \in C_k$ lies in $\operatorname{img} d_{k+1}$, then we have $\eta = d_{k+1}(\zeta)$ for some ζ in C_{k+1}. Once again, Definition 4.6 gives us,

$$\phi_k(\eta) = \phi_k \circ d_{k+1}(\zeta) = d'_{k+1} \circ \phi_{k+1}(\zeta),$$

whence $\phi_k(\eta)$ lies in $\operatorname{img} d_{k+1}$. Thus, for each ξ in $\ker d_k$, our map $\mathbf{H}_k\phi$ sends $\xi + \operatorname{img} d_{k+1}$ to $\phi(\xi) + \operatorname{img} d'_{k+1}$, with a guarantee that $\phi(\xi)$ lies in $\ker d'_k$. □

Combining Proposition 4.5 with Proposition 4.7, we see that every simplicial map $f : K \to L$ indeed produces a well-defined linear map $\mathbf{H}_k(K;\mathbb{F}) \to \mathbf{H}_k(L;\mathbb{F})$ for each dimension $k \geq 0$. In order to avoid writing the monstrosity $\mathbf{H}_k\mathbf{C}f$ every time we want to mention this *induced map*, we will abbreviate it to $\mathbf{H}_k f : \mathbf{H}_k(K;\mathbb{F}) \to \mathbf{H}_k(L;\mathbb{F})$. It is not difficult to confirm that when f is the identity simplicial map $K \to K$, its induced map is the identity on $\mathbf{H}_k(K;\mathbb{F})$. The following result (which forms one of the exercises to this Chapter) takes a bit more work.

PROPOSITION 4.8. *Given chain maps $\phi_\bullet : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ and $\psi : (C'_\bullet, d'_\bullet) \to (C''_\bullet, d''_\bullet)$, we have*

$$\mathbf{H}_k(\psi \circ \phi) = \mathbf{H}_k\psi \circ \mathbf{H}_k\phi$$

for each dimension $k \geq 0$

Applying the above result to the special case where our chain maps are induced by simplicial maps (i.e., $\phi_\bullet = \mathbf{C}_\bullet f$ and $\psi_\bullet = \mathbf{C}_\bullet g$ for some $f : K \to L$ and $g : L \to M$) completes the proof of Theorem 4.3.

We now have the ability to study not just the homology groups of simplicial complexes but also linear maps of homology groups induced by simplicial maps (and more generally, chain maps); we will now examine various salient properties of such maps. A chain map $\phi_\bullet$ is called an isomorphism if each $\phi_k : C_k \to C'_k$ is an isomorphism from C_k to C'_k — for such a ϕ the induced maps $\mathbf{H}_\bullet\phi$ are also isomorphisms. But in general $\mathbf{H}_\bullet\phi$ can be an isomorphism even if $\phi_\bullet$ is not.

DEFINITION 4.9. A chain map $\phi_\bullet : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ is called a **quasi-isomorphism** if the induced map $\mathbf{H}_k\phi : \mathbf{H}_k(C_\bullet, d_\bullet) \to \mathbf{H}_k(C'_\bullet, d'_\bullet)$ is an isomorphism for each dimension $k \geq 0$.

In sharp contrast to testing whether a simplicial map induces homotopy equivalence or not, testing whether it induces a quasi-isomorphic chain map (and hence, isomorphisms of homology groups) is algorithmic and machine-computable.

REMARK 4.10. Consider a chain map $\phi : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$; if the dimensions of C_k and C'_k are finite, then the computation of $\mathbf{H}_k\phi$ can be accomplished via the following linear algebraic procedure:

(1) Extract basis vectors B and B' for $\mathbf{H}_k(C_\bullet, d_\bullet)$ and $\mathbf{H}_k(C'_\bullet, d'_\bullet)$ via Proposition 3.15.
(2) For each basis chain b in B, write $\phi_k(b)$ as a linear combination of the basis chains of B':

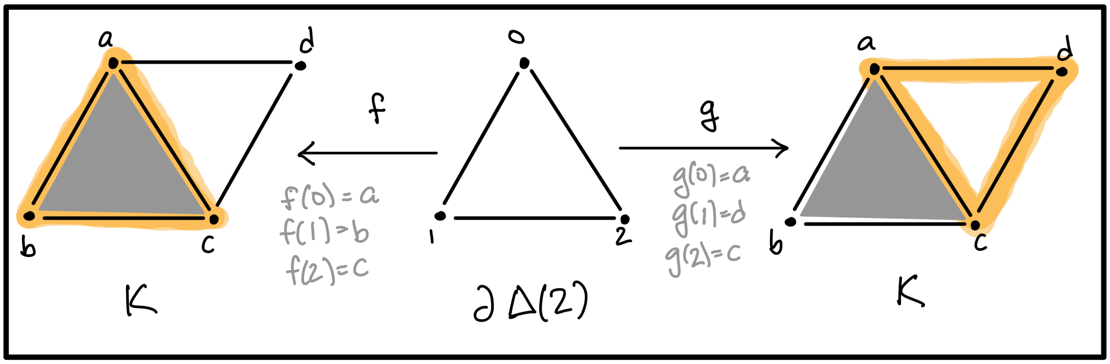
$$\phi_k(b) = \sum_{b'} \alpha_{b,b'} \cdot b',$$

where each $\alpha_{b,b'}$ lies in the coefficient field $\mathbb{F}$. These α coefficients can be determined for all b at once by row-reducing the augmented matrix $[B'_k \mid \phi_k(B)]$.
(3) The coefficients $\{\alpha_{b,b'} \mid b \in B \text{ and } b' \in B'\}$ form a matrix $\mathbf{H}_k(C_\bullet, d_\bullet) \to \mathbf{H}_k(C'_\bullet, d'_\bullet)$; this matrix represents our linear map $\mathbf{H}_k\phi$ in terms of the bases B and B'.

Computability issues aside, induced maps on homology can be somewhat subtle.

EXAMPLE 4.11. The figure below illustrates two simplicial maps f, g from the hollow 2-simplex $\partial\Delta(2)$ to another simplicial complex K. The homology groups of K and $\partial\Delta(2)$ are isomorphic as rational vector spaces, i.e.,

$$\mathbf{H}_k(\partial\Delta(2);\mathbb{Q}) \simeq \mathbf{H}_k(K;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k \in \{0,1\}, \\ 0 & \text{otherwise .} \end{cases}$$



The chain map $\mathbf{C}_\bullet g$ is a quasi-isomorphism whereas $\mathbf{C}_\bullet f$ is not.

4.4 Chain Homotopy

There is a purely algebraic version of homotopy equivalence designed to work directly with chain complexes (rather than topological spaces). As usual, the first step is to define an equivalence relation between the set of all chain maps between a fixed pair of chain complexes.

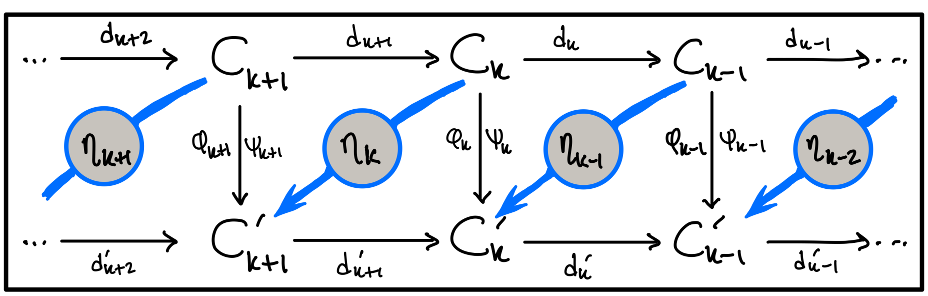
DEFINITION 4.12. A **chain homotopy** $\eta_\bullet$ between chain maps $\phi_\bullet, \psi_\bullet : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ is a collection of $\mathbb{F}$-linear maps $\eta_k : C_k \to C'_{k+1}$ which satisfy

$$\phi_k - \psi_k = \eta_{k-1} \circ d_k + d'_{k+1} \circ \eta_k$$

for each $k \geq 0$.

We write $\eta_\bullet : \phi_\bullet \Rightarrow \psi_\bullet$ to indicate that $\eta_\bullet$ is a chain homotopy as defined above; and the maps $\phi_\bullet$ and $\psi_\bullet$ are said to be **chain homotopic** whenever such an $\eta_\bullet$ exists. Chain homotopy is an equivalence relation on the set of all chain maps between a fixed pair of chain complexes.

REMARK 4.13. It is important to note that the linear maps η_k are *not* required to satisfy any relations beyond the ones in the preceding defintion — in particular, they do not have to commute with d, d', ϕ or ψ. Even so, it is good to see how they fit within the commuting staircase diagrams that contain ϕ and ψ:



The following result highlights the utility of chain homotopy.

PROPOSITION 4.14. *If $\phi_\bullet, \psi_\bullet : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ are chain homotopic, then their induced maps on homology coincide, i.e.,*

$$\mathbf{H}_k \phi = \mathbf{H}_k \psi$$

for each dimension $k \geq 0$.

PROOF. Let $\eta_\bullet : \phi_\bullet \Rightarrow \psi_\bullet$ be a chain homotopy. For any chain $\gamma \in \ker d_k$, Definition 4.12 gives us

$$\phi_k(\gamma) - \psi_k(\gamma) = \eta_{k-1} \circ d_k(\gamma) + d'_{k+1} \circ \eta_k(\gamma).$$

But $d_k(\gamma) = 0$, so the first term on the right side disappears and the difference $\phi_k(\gamma) - \psi_k(\gamma)$ equals $d'_{k+1} \circ \eta_k(\gamma)$, which evidently lies in img d'_{k+1}. Thus, this difference is always a k-boundary, which is undetectable by homology. □

Chain homotopies are to chain maps what homotopies (as in Definition 2.1) are to continuous maps: they provide an indirect method for establishing that two chain complexes $(C_\bullet, d_\bullet)$ and $(C'_\bullet, d'_\bullet)$ are related by a quasi-isomorphism. The good news is that this method largely circumvents the tedious algebraic manipulations of Remark 4.10 and Proposition 3.15. But the bad news is that in order to avail of this method, we require not only a backwards chain map $\psi_\bullet : (C'_\bullet, d'_\bullet) \to (C_\bullet, d_\bullet)$ but also a pair of chain homotopies, described below.

DEFINITION 4.15. A pair of chain complexes is said to be **chain homotopy equivalent** if there are two chain maps

$$\phi_\bullet : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet) \text{ and } \psi_\bullet : (C'_\bullet, d'_\bullet) \to (C_\bullet, d_\bullet)$$

along with chain homotopies

$$\eta_\bullet : 1_{(C_\bullet, d_\bullet)} \Rightarrow \psi_\bullet \circ \phi_\bullet \text{ and } \eta'_\bullet : \phi_\bullet \circ \psi_\bullet \Rightarrow 1_{(C'_\bullet, d'_\bullet)}.$$

Here $1_{(C_\bullet, d_\bullet)}$ is the identity chain map of $(C_\bullet, d_\bullet)$, etc.

It follows immediately from Proposition 4.14 that if two chain complexes are chain homotopy equivalent, then they must be related by quasi-isomorphisms and hence have isomorphic homology groups.

EXAMPLE 4.16. The cone Cone(K) over any simplicial complex K (see Definition 1.19) has homology groups isomorphic to those of $\Delta(0)$, namely:

$$\mathbf{H}_k(\text{Cone}(K); \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k > 0. \end{cases}$$

To see this, let $f : \text{Cone}(K) \to \Delta(0)$ be the simplicial map sending every vertex of the cone to the unique vertex 0, and let $g : \Delta(0) \to \text{Cone}(K)$ be the simplicial map sending 0 to the special vertex v_* which lies in Cone$(K) - K$. Now the composite $f \circ g$ equals the identity on $\Delta(0)$, and the other composite $g \circ f$ sends every vertex of Cone(K) to v_*. It remains to construct a chain homotopy from the identity chain map on $\mathbf{C}_\bullet(\text{Cone}(K))$ to the composite $\mathbf{C}_\bullet(g \circ f)$. This will be accomplished in one of the exercises to this Chapter.

4.5 Exact Sequences and the Snake Lemma

Our study of homotopy equivalence benefited greatly from a thorough analysis of contractible spaces, i.e., the spaces which have the simplest possible homotopy type. For analogous reasons, we ask which chain complexes have trivial homology.

DEFINITION 4.17. A sequence of vector spaces and linear maps

$$\cdots \xrightarrow{a_{k+2}} V_{k+1} \xrightarrow{a_{k+1}} V_k \xrightarrow{a_k} V_{k-1} \xrightarrow{a_{k-1}} \cdots$$

is said to be **exact at** k if ker a_k equals img a_{k+1} as subpaces of V_k. The entire sequence is called **exact** if it is exact at every $k \in \mathbb{N}$.

A casual glance at Definition 3.9 will confirm that every exact sequence is a chain complex, and another brief look at Definition 3.11 reveals that exact sequences are precisely those chain complexes whose homology group is trivial in every dimension $k \geq 0$. We call an exact sequence **short** if all but three of the V_i (let's say V_0, V_1 and V_2 without loss of generality) are required to be trivial. Short exactness relates to standard notions in linear algebra, for instance:

(1) $0 \to V_1 \to V_0$ is exact at $k = 1$, iff $V_1 \to V_0$ is injective,
(2) $V_2 \to V_1 \to 0$ is exact at $k = 1$ iff $V_2 \to V_1$ is surjective,
(3) $0 \to V_2 \to V_1 \to 0$ is exact at $k \in \{1, 2\}$ iff $V_2 \to V_1$ is an isomorphism, and
(4) $0 \to V_2 \to V_1 \to V_0 \to 0$ is exact iff $V_1 = V_0 \oplus V_2$.

The first three of these statements hold in broader contexts (i.e, we can replace the vector spaces by abelian groups) whereas the last one is specific to vector spaces. The definition of a short exact sequence also extends verbatim to chain complexes.

DEFINITION 4.18. A **short exact sequence of chain complexes** consists of three chain complexes and two chain maps arranged as follows:

$$(C_\bullet, d_\bullet) \xrightarrow{\phi_\bullet} (C'_\bullet, d'_\bullet) \xrightarrow{\psi_\bullet} (C''_\bullet, d''_\bullet),$$

with the additional requirement that for each $k \geq 0$ the chain groups

$$0 \longrightarrow C_k \xrightarrow{\phi_k} C'_k \xrightarrow{\psi_k} C''_k \longrightarrow 0$$

form a short exact sequence of $\mathbb{F}$-vector spaces.

The following lemma is by far the most important result in this Chapter; it forms the first of many miracles in the field of *homological algebra*.

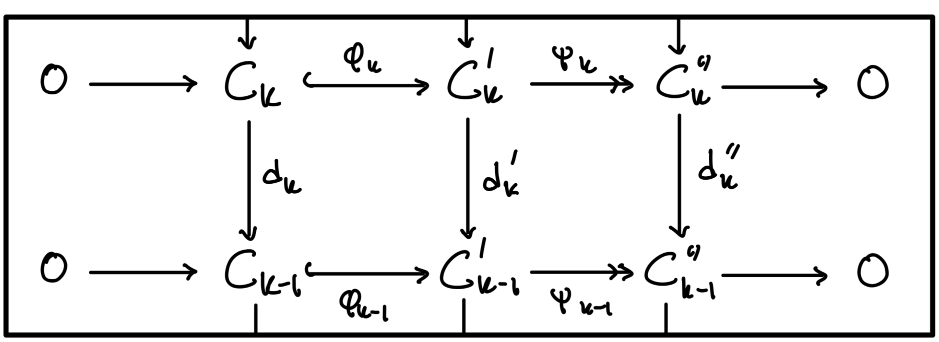
LEMMA 4.19. **[The Snake lemma.]** *For each short exact sequence of chain complexes*

$$(C_\bullet, d_\bullet) \xrightarrow{\phi_\bullet} (C'_\bullet, d'_\bullet) \xrightarrow{\psi_\bullet} (C''_\bullet, d''_\bullet),$$

there exists a family of linear maps $D_k : \mathbf{H}_k(C''_\bullet, d''_\bullet) \to \mathbf{H}_{k-1}(C_\bullet, d_\bullet)$ *which fit into an exact sequence of homology groups:*

$$\cdots \xrightarrow{D_{k-1}} \mathbf{H}_k(C_\bullet, d_\bullet) \xrightarrow{\mathbf{H}_k\phi} \mathbf{H}_k(C'_\bullet, d'_\bullet) \xrightarrow{\mathbf{H}_k\psi} \mathbf{H}_k(C''_\bullet, d''_\bullet) \xrightarrow{D_k} \mathbf{H}_{k-1}(C_\bullet, d_\bullet) \xrightarrow{\mathbf{H}_{k-1}\phi} \cdots$$

The collection of linear maps $\{D_k \mid k \geq 1\}$ is called the **connecting homomorphism** of the given short exact sequence. The full proof of this lemma is a tedious affair, and tends to be far from enlightening. We will say just enough about it here to explain the serpentine etymology. To build D_k, one starts with the piece of the short exact sequence connecting dimensions k and $k-1$:

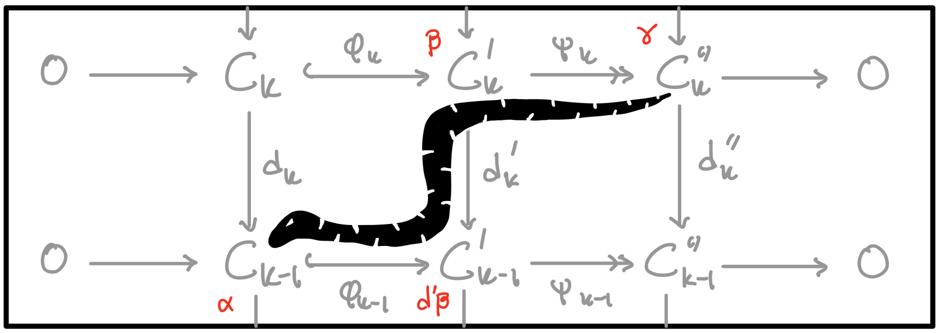


Since both rows are exact by Definition 4.18, the ϕ maps are injective while the ψ maps are surjective. We'd like D_k to send elements of the homology group $\mathbf{H}_k(C''_\bullet, d''_\bullet)$ to elements of

$\mathbf{H}_{k-1}(C_\bullet, d_\bullet)$, so it makes sense to start with the upper-right corner of this diagram. There are four basic steps in the construction:

(1) Choose any γ lying in $\ker d''_k \subset C''_k$.
(2) By surjectivity, there is some β in C'_k satisfying $\psi_k(\beta) = \gamma$.
(3) Since ψ is a chain map, Definition 4.6 gives us
$$\psi_{k-1} \circ d'_k(\beta) = d''_k \circ \psi_k(\beta) = d''_k(\gamma) = 0;$$
thus, $d'_k(\beta)$ lies in $\ker \psi_{k-1}$.
(4) By exactness of the bottom row, this kernel equals the image of ϕ_{k-1}, so there is some α in C_{k-1} satisfying $\phi_{k-1}(\alpha) = d'_k(\beta)$.

One defines $D_k(\gamma) = \alpha$ as the desired map. The promised snake materializes when we trace the path taken in our short exact sequence $\gamma \mapsto \beta \mapsto d'_k\beta \mapsto \alpha$:



The argument is far from complete: one must show that D_k defines a well-defined map on homology independent of our choice of β, that α lies in $\ker d_{k-1}$, and that the sequence involving $\mathbf{H}_k\phi, \mathbf{H}_k\psi$ and D_k is exact. We will only perform the second check here:
$$\begin{aligned}
\phi_{k-2} \circ d_{k-1}(\alpha) &= d'_{k-1} \circ \phi_{k-1}(\alpha) && \text{by Definition 4.6,}\\
&= d'_{k-1} \circ d'_k(\beta) && \text{since } \phi_{k-1}(\alpha) = d'_k(\beta),\\
&= 0 && \text{since } (C'_\bullet, d'_\bullet) \text{ is a chain complex.}
\end{aligned}$$
But ϕ_{k-2} is injective by exactness, so $d_{k-1}(\alpha) = 0$ as desired.

4.6 Pairs and Relative Homology

One of the first applications of Lemma 4.19 is the ability to relate the homology groups of a simplicial complex K, a subcomplex $L \subset K$ and the topological quotient $|K|/|L|$. This quotient does not form a simplicial complex in any natural way, but we are still able to define its homology by building an appropriate quotient chain complex as follows. Each chain group $\mathbf{C}_k(L)$ is a subspace of the corresponding $\mathbf{C}_k(K)$ by Definition 3.6. And since tje faces of every simplex in L themselves lie in L by the subcomplex property, the restriction of ∂_k^K to $\mathbf{C}_k(L)$ coincides with ∂_k^L by Definition 3.7. Thus, ∂_k^K induces a well-defined map of quotient spaces, which we denote
$$\partial_k^{K,L} : \mathbf{C}_k(K)/\mathbf{C}_k(L) \to \mathbf{C}_{k-1}(K)/\mathbf{C}_{k-1}(L).$$
Since $\partial_\bullet^K$ is a boundary operator, it follows that $\partial_k^{K,L} \circ \partial_{k+1}^{K,L} = 0$.

Definition 4.20. Let $L \subset K$ be a pair of simplicial complexes; the **relative homology groups** $\mathbf{H}_k(K, L)$ are defined to be the homology groups of the chain complex defined as follows: its chain groups are

$$\mathbf{C}_k(K,L) = \mathbf{C}_k(K)/\mathbf{C}_k(L),$$

and the boundary operators are $\partial_k^{K,L} : \mathbf{C}_k(K,L) \to \mathbf{C}_{k-1}(K,L)$.

The Snake lemma enters the picture because whenever $L \subset K$ is a subcomplex, we have an apparent short exact sequence of chain complexes

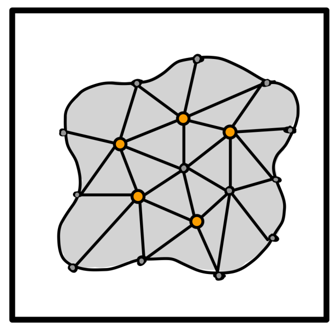
$$\left(\mathbf{C}_\bullet(L), \partial_\bullet^L\right) \xhookrightarrow{\iota_\bullet} \left(\mathbf{C}_\bullet(K), \partial_\bullet^K\right) \xrightarrow{\pi_\bullet} \left(\mathbf{C}_\bullet(K,L), \partial_\bullet^{K,L}\right).$$

Here the chain map $\iota_\bullet$ is given by inclusions of subspaces while the chain map $\pi_\bullet$ is given by projections to quotient spaces. Applying Lemma 4.19 to this short exact sequence, we obtain a connecting homomorphism $D_k : \mathbf{H}_k(K,L) \to \mathbf{H}_{k-1}(L)$ and hence the following exact sequence relating homology groups.

DEFINITION 4.21. The **exact sequence of the pair** $L \subset K$ of simplicial complexes is given by

$$\cdots \xrightarrow{D_{k+1}} \mathbf{H}_k(L) \xrightarrow{\mathbf{H}_k\iota} \mathbf{H}_k(K) \xrightarrow{\mathbf{H}_k\pi} \mathbf{H}_k(K,L) \xrightarrow{D_k} \mathbf{H}_{k-1}(L) \xrightarrow{\mathbf{H}_{k-1}\iota} \cdots$$

The exact sequence of a pair is a wonderful tool for computing relative homology groups $\mathbf{H}_\bullet(K,L)$ using prior knowledge of $\mathbf{H}_k(K)$ and $\mathbf{H}_k(L)$. Consider, for instance, the scenario where K is any simplicial complex whose realization $|K|$ is homeomorphic to the 2-dimensional disk, while the subcomplex $L \subset K$ consists of n interior vertices — the case $n = 5$ has been illustrated. Building the chain complex $\mathbf{C}_\bullet(K,L)$ which yields the relative homology is quite a chore, but the exact sequence of a pair works remarkably well. We know (or can compute, if asked) that the homology of K agrees with that of $\Delta(2)$, whereas L consists of n disjoint copies of $\Delta(0)$. Putting all this known information about K and L together, we have:



$$\mathbf{H}_k(K;\mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k > 0 \end{cases} \quad \text{and} \quad \mathbf{H}_k(L;\mathbb{F}) = \begin{cases} \mathbb{F}^n & k = 0 \\ 0 & k > 0 \end{cases}.$$

All the non-trivial bits of the exact sequence of this pair concentrate in the lower dimensions — here is the relevant piece of the sequence:

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{H}_1(K,L) \xrightarrow{D_0} \mathbf{H}_0(L) \xrightarrow{\mathbf{H}_0\iota} \mathbf{H}_0(K) \xrightarrow{\mathbf{H}_0\pi} \mathbf{H}_0(K,L) \longrightarrow 0$$

Everything depends on the rank of the map $\mathbf{H}_0\iota$ which is induced on 0-th homology by the inclusion of L into K. It is straightforward to check that this is not the zero map, and hence has rank 1. Now exactness of this sequence immediately forces the rank of D_0 to be $n-1$ and the rank of $\mathbf{H}_0\pi$ to be 0. But D_0 is injective and $\mathbf{H}_0\pi$ is surjective (because of the leading and trailing 0's plus exactness), which gives

$$\mathbf{H}_k(K,L) = \begin{cases} \mathbb{F}^{n-1} & k = 1 \\ 0 & k \neq 1 \end{cases}.$$

REMARK 4.22. The relative homology of a pair $L \subset K$ generalizes ordinary simplicial homology of K if we allow ourselves the luxury of setting $L = \varnothing$; in this case the chain groups $\mathbf{C}_\bullet(K)$ and $\mathbf{C}_\bullet(K,L)$ are equal, as are the boundary operators. On the other hand, the relative homology of a pair is further generalized by that of a **triple** $M \subset L \subset K$ of simplicial complexes. Here the short exact sequence of interest is

$$(\mathbf{C}_\bullet(L,M), \partial_\bullet^{L,M}) \xhookrightarrow{\iota_\bullet} (\mathbf{C}_\bullet(K,M), \partial_\bullet^{K,M}) \xrightarrow{\pi_\bullet} (\mathbf{C}_\bullet(K,L), \partial_\bullet^{K,L}),$$

Once again, the chain map $\iota_\bullet$ is an inclusion while the map π is a projection; the subcomplex property is necessary to get well-defined boundary operators of these relative chain complexes (as it was in Definition 4.20). The connecting homomorphisms $D_k : \mathbf{H}_k(K,L) \to \mathbf{H}_{k-1}(L,M)$ guaranteed by Lemma 4.19 fit into an exact sequence with $\mathbf{H}_\bullet\iota$ and $\mathbf{H}_\bullet\pi$.

4.7 The Mayer-Vietoris Sequence

A second enormously useful application of the Snake lemma is that it confers the ability to compute homology of a complicated simplicial complex K in terms of a decomposition into two (hopefully simpler) subcomplexes. Assume that L and M are subcomplexes of K satisfying $K = L \cup M$, and let's agree to write their intersection $L \cap M$ – which must also be a subcomplex of K – as I. There are now four chain complexes and four chain maps (all inclusions) to keep track of; these fit into the following diamond:

$$\begin{array}{ccccc} & & (\mathbf{C}_\bullet(I), \partial_\bullet^I) & & \\ & \swarrow & & \searrow & \\ (\mathbf{C}_\bullet(L), \partial_\bullet^L) & & & & (\mathbf{C}_\bullet(M), \partial_\bullet^M) \\ & \searrow & & \swarrow & \\ & & (\mathbf{C}_\bullet(K), \partial_\bullet^K) & & \end{array}$$

Both paths from the top to the bottom give the same chain map (the one which includes chains of I into chains of K); thus our diamond commutes in the category $\mathbf{Chain}_{\mathbb{F}}$. The crucial idea here is to generate a short exact sequence by combining the two chain complexes of the middle row into a single one.

The *direct sum* of $(\mathbf{C}_\bullet(L), \partial_\bullet^L)$ and $(\mathbf{C}_\bullet(M), \partial_\bullet^M)$ is the new chain complex defined as follows: in each dimension $k \geq 0$, it has

$$\text{chain groups } \mathbf{C}_k(L) \oplus \mathbf{C}_k(M) \text{ and boundary operator } \begin{bmatrix} \partial_k^L & 0 \\ 0 & \partial_k^M \end{bmatrix}.$$

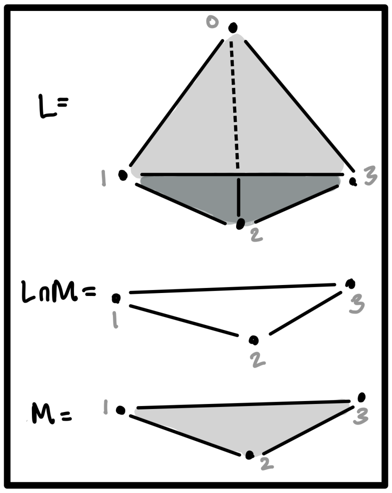
The k-th homology group of this direct sum is $\mathbf{H}_k(L) \oplus \mathbf{H}_k(M)$. More interestingly, there is an injective chain map $\iota_k : \mathbf{C}_k(I) \to \mathbf{C}_k(M) \oplus \mathbf{C}_k(L)$ which sends every ξ to the pair (ξ, ξ). There is also a second chain map $\pi_k : \mathbf{C}_k(M) \oplus \mathbf{C}_k(L) \to \mathbf{C}_k(K)$ that sends each pair (α, β) to the difference $(\beta - \alpha)$. This map $\pi_\bullet$ is evidently surjective because $K = L \cup M$; thus, we obtain

$$(\mathbf{C}_\bullet(I), \partial_\bullet^I) \xhookrightarrow{\iota_\bullet} (\mathbf{C}_\bullet(L), \partial_\bullet^L) \oplus (\mathbf{C}_\bullet(M), \partial_\bullet^M) \xrightarrow{\pi_\bullet} (\mathbf{C}_\bullet(K), \partial_\bullet^K).$$

This turns out to be a short exact sequence: note that $(\alpha, \beta) \in \mathbf{C}_k(L) \cap \mathbf{C}_k(M)$ lies in $\ker \pi_k$ if and only if $\alpha = \beta$. But this equality holds if and only if the chain α lies in the intersection $\mathbf{C}_k(I) = \mathbf{C}_k(L) \cap \mathbf{C}_k(M)$, whence (α, α) lies in $\operatorname{img} \iota_k$. Having obtained a short exact sequence of chain complexes, we appeal once more to the Snake lemma and obtain a connecting homomorphism $D_k : \mathbf{H}_k(K) \to \mathbf{H}_{k-1}(I)$.

DEFINITION 4.23. Let $K = L \cup M$ be a decomposition of the simplicial complex K into two subcomplexes L and M whose intersection is denoted I. The **Mayer-Vietoris exact sequence** associated to this partition is given by

$$\cdots \xrightarrow{D_{k-1}} \mathbf{H}_k(I) \xrightarrow{\mathbf{H}_k\iota} \mathbf{H}_k(L) \oplus \mathbf{H}_k(M) \xrightarrow{\mathbf{H}_k\pi} \mathbf{H}_k(K) \xrightarrow{D_k} \mathbf{H}_{k-1}(I) \xrightarrow{\mathbf{H}_{k-1}\iota} \cdots$$



This exact sequence is particularly effective when combined with inductive arguments — we can use it to compute the i-th homology group of every hollow k-simplex $\partial\Delta(k)$ for $i > 1$. Consider the decomposition $\partial\Delta(k) = L \cup M$ where L is the closed star of the vertex 0 (see Definition 1.17) while M consists of the simplex $\{1, 2, \ldots, k\}$ along with all its faces. This decomposition is illustrated for $k = 3$ here. Note also that the intersection $L \cap M$ is the hollow simplex of one lower dimension, i.e., $\partial\Delta(k-1)$.

Now we claim that both L and M have the same homology as $\Delta(0)$. First note that L is clearly a cone over $\partial\Delta(k-1)$, so the conclusion follows from Example 4.16. And M is a solid k-simplex, which is a cone over a solid $(k-1)$-simplex, so once again Example 4.16 does the job. Consequently, the homology groups $\mathbf{H}_i(L)$ and $\mathbf{H}_i(M)$ are trivial for all $i > 0$, and hence so is their direct sum. So for each $i > 1$, we obtain the following snippet of the Mayer-Vietoris exact sequence:

$$0 \longrightarrow \mathbf{H}_i(\partial\Delta(k)) \xrightarrow{D_i} \mathbf{H}_{i-1}(\partial\Delta(k-1)) \longrightarrow 0$$

Exactness forces D_i to be an isomorphism for all $i > 1$, so it suffices to calculate the homology groups $\mathbf{H}_i(\partial\Delta(2); \mathbb{F})$ as a base case; we did this in Example 3.12, and can safely conclude that for $i > 0$ we have:

$$\mathbf{H}_i(\partial\Delta(k); \mathbb{F}) = \begin{cases} \mathbb{F} & i = k-1 \\ 0 & \text{otherwise.} \end{cases}$$

A separate (and somewhat easier) argument must be used to compute $\mathbf{H}_0(\partial\Delta(k))$.

4.8 Bonus: Homotopy Invariance

Theorem 4.24 below is vital from both a theoretical and practical perspective; its proof requires techniques which are outside our scope at the moment, but the ability to understand and apply it will be quite beneficial when working with homology.

As mentioned at the beginning of Chapter 3, the Euler characteristic inherits its homotopy invariance from homology.

Theorem 4.24. *Let K and L be simplicial complexes. For any choice of coefficient field $\mathbb{F}$,*

(1) *if $f, g : K \to L$ are homotopic simplicial maps, then $\mathbf{H}_k f = \mathbf{H}_k g$ for every $k \geq 0$; and,*
(2) *if K and L are homotopy equivalent, then $\mathbf{H}_k(K)$ is isomorphic to $\mathbf{H}_k(L)$ for every $k \geq 0$.*

The second assertion follows from the first one if we use simplicial approximation (see Theorem 2.15). The basic idea is to start with topology and gradually descend to algebra: Assume that $\theta : |K| \times [0,1] \to |L|$ is a homotopy from $|f|$ to $|g|$. The first order of business is to build a simplicial complex homeomorphic to $|K| \times [0,1]$ — this is rendered difficult by the fact that in general the product of a simplex with $[0,1]$ is not itself a simplex in any natural way. Fortunately, such a product can be triangulated into a union of simplices, and putting these together produces a simplicial complex $P(K)$ whose realization is homeomorphic to $|K| \times [0,1]$. Using this homeomorphism, one approximates the homotopy θ as a simplicial map $\mathbf{Sd}^n\, P(K) \to L$ (where **Sd** stands for barycentric subdivision). This approximated version of θ then descends to a chain homotopy from $\mathbf{C}_\bullet f$ to $\mathbf{C}_\bullet g$. An appeal to Proposition 4.14 completes the argument.

Exercises

EXERCISE 4.1. Given functors $\boldsymbol{F} : \mathscr{C} \to \mathscr{C}'$ and $\boldsymbol{G} : \mathscr{C}' \to \mathscr{C}''$, define their composite $\boldsymbol{G} \circ \boldsymbol{F}$ and show that this is a functor $\mathscr{C} \to \mathscr{C}''$.

EXERCISE 4.2. Show that the collection of all simplicial complexes and simplicial maps satisfies the axioms of a category **SC**.

EXERCISE 4.3. Consider two chain maps $\phi : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ and $\psi : (C'_\bullet, d'_\bullet) \to (C''_\bullet, d''_\bullet)$. Show that the collection of maps $\psi_k \circ \phi_k : C_k \to C''_k$ prescribe a chain map from $(C_\bullet, d_\bullet)$ to $(C''_\bullet, d''_\bullet)$. Thus, chain maps are morphisms in the category $\mathbf{Chain}_{\mathbb{F}}$ of chain complexes over $\mathbb{F}$.

EXERCISE 4.4. Given simplicial maps $f : K \to L$ and $g : L \to M$, show that $\mathbf{C}_k(g \circ f)$ equals $\mathbf{C}_k(g) \circ \mathbf{C}_k(f)$. This shows that $\mathbf{C}$ is a functor from the **SC** of Exercise 4.2 to the category $\mathbf{Chain}_{\mathbb{F}}$ of Exercise 4.3.

EXERCISE 4.5. Write down a proof of Proposition 4.8.

EXERCISE 4.6. Verify the assertions of Example 4.11.

EXERCISE 4.7. In the setting of Example 4.11, consider the simplicial map $h : K \to \partial\Delta(2)$ that sends vertex a to 0, vertex d to 1 and vertex c to 2. Show that $\mathbf{H}_k h$ is an inverse to $\mathbf{H}_k g$ for every $k \geq 1$. (Note that h and g themselves are not inverse to each other as chain maps!)

EXERCISE 4.8. Prove that chain homotopy is an equivalence relation on the set of all chain maps $(C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$.

EXERCISE 4.9. Using $\mathbb{F} = \mathbb{Z}/2$ coefficients, complete the argument of Example 4.16 as follows. Define the linear maps $\eta_k : \mathbf{C}_k(\mathrm{Cone}(K)) \to \mathbf{C}_{k+1}(\mathrm{Cone}(K))$ that sends each basis k-simplex σ to

$$\eta_k(\sigma) = \begin{cases} \sigma \cup \{v_*\} & \sigma \in K \\ 0 & \sigma \in \mathrm{Cone}(K) - K. \end{cases}$$

Show that $\eta_\bullet$ prescribes a chain homotopy between the chain map $\phi_\bullet := \mathbf{C}_\bullet(g \circ f)$ and the identity chain map. [Hint: let d be the boundary operator for the simplicial chain complex of $\mathrm{Cone}(K)$. Over $\mathbb{F} = \mathbb{Z}/2$ we don't have to concern ourselves with any minus signs, and it suffices to show that $\sigma + \phi(\sigma) = d \circ \eta(\sigma) + \eta \circ d(\sigma)$ for each simplex σ. Start with $\dim \sigma = 0$ and induct upwards along dimension.]

EXERCISE 4.10. For each $k \geq 1$, compute the relative homology group $\mathbf{H}_k(\Delta(k), \partial\Delta(k))$.

EXERCISE 4.11. Let K and L be simplicial complexes. Identify a vertex v of K with a vertex w of L to form a new simplicial complex $K \vee L$. Prove that $\mathbf{H}_k(K \vee L) = H_k(K) \oplus H_k(L)$ for all $k > 0$ [Hint: Mayer-Vietoris].